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Consider a set of N systems and an arbitrary interaction Hamiltonian H that couples them. We investigate the use of local operations and classical communication (LOCC), together with the Hamiltonian H, to simulate a unitary evolution of the N systems according to some other Hamiltonian H'. First, we show that the most general simulation using H and LOCC can be also achieved by just interspersing the evolution of H with local unitary manipulations of each system and a corresponding local ancilla (in a so-called LU+anc protocol). Thus, LOCC protocols and LU+anc protocols are equivalent. Second, we show that both for the case of two d-level systems ($d > 2$), or for that of any setting with more than two systems ($N \geq 3$), LU+anc protocols are more powerful than LU protocols —using local unitaries without ancillas. Finally, we use results of majorization theory to explicitly solve the problem of optimal simulation of two-qubit Hamiltonians using LU (and thus also LOCC).

03.67.-a, 03.65.Bz, 03.65.Ca, 03.67.Hk

I. INTRODUCTION

The problem of using a given nonlocal Hamiltonian H and some type of local resources to simulate another nonlocal Hamiltonian H' has attracted the attention of several authors very recently [1–4]. Hamiltonian simulation has immediate practical applications in quantum control theory, for it allows to achieve any desired interaction H' between systems whose natural interaction is given by H [5]. It also sets a frame for the parameterization of the nonlocal resources contained in multiparticle Hamiltonians, very much in the line of thought pursued to quantify the entanglement of quantum states.

It has been recently shown [4] that the efficiency with which Hamiltonian H, together with local operations (LO), simulates Hamiltonian H' can be used as a criterion to endow the set of Hamiltonians with a (pseudo) partial order structure, that allows to compare the nonlocal capabilities of H and H'. For two-qubit Hamiltonians this structure was developed explicitly, providing the necessary and sufficient conditions for H to be able to simulate H' *efficiently* for infinitesimal times, that is, the conditions under which the use of H for time t allows to simulate H' for the same time t, for $t \to 0$. Equivalently, it was shown how to *optimally* simulate H' with H, in the sense of achieving the maximal simulation ratio t'/t (again in the limit $t \to 0$), where t is the time of interaction H that it takes to simulate interaction H' for a time t'. These results were obtained through a detailed geometrical analysis of a three dimensional polytope.

A. Classical communication in Hamiltonian simulation

In this paper we add an important resource to the simulation of interactions that was missing in previous studies. Namely, we consider the problem of simulating Hamiltonians when not only LO on the interacting systems, but also classical communication (CC) between the parties that locally manipulate them, are allowed.

Enlarging the set of allowed manipulations from LO to LOCC implies that the parties can communicate the results of their local measurements and modify subsequent manipulations accordingly. Notice that, as a matter of fact, even within the more restrictive set of LO, the parties can already use the available interaction H to communicate classical bits. This, however, has a high price in terms of simulation time. Allowing for "free" CC in the simulation process means that we are finally able to focus our quantitative studies only on those aspects of the interaction H that are genuinely quantum. Marginally, this also enhances the parallelism with the studies of entangled states, where the set of LOCC transformations constitute a most natural frame.

B. Results

This paper contains the following three main results concerning the simulation of Hamiltonian evolutions for infinitesimal times:

(i) LOCC simulation protocols can be reduced to LU+anc simulation protocols. That is, for N-particle Hamiltonian interactions H and H', any protocol that simulates H' using H and LOCC can be replaced, without changing its time efficiency, with a protocol involving only H and local unitary transformations. Each local unitary transformation is performed on the system and a local ancilla.

(ii) Apart from exceptional cases such as that of two-qubit Hamiltonians [4] —in which any LU+anc protocol can be replaced with a protocol using only LU on each qubit—, the use of ancillas is, in general, advantageous. This is proven by constructing explicit examples of protocols where ancillas are used to obtain simulations that

cannot be achieved without them, both in the case of two d-level systems ($d > 2$) and in the case of $N > 2$ systems.

(*iii*) For two-qubit Hamiltonians we use results of majorization theory to recover the optimality results presented in [4], and thereby solving, in view of result (*i*), the problem of optimal two-qubit Hamiltonian simulation using LOCC.

C. Preliminaries

Before we start presenting the above results some background material needs to be introduced [4].

1. Hamiltonian simulation and classes of operations

Recall that the aim of Hamiltonian simulation is, given a set of systems that interact according to Hamiltonian H for time t and a class C of allowed operations, to be able to produce an evolution $e^{-iH't'}$ for the systems, where H' and t' are the simulated Hamiltonian and the simulated time. [We take $\hbar \equiv 1$ along the paper].

One can consider several classes of operations to assist the simulation, including LU, LU+anc, LO and LOCC. We assume them to be a cheap resource, so that optimality over simulation protocols is defined only in terms of the ratio t'/t, that is, in terms of how much time t' of evolution according to H' can be produced by using H for a time t. Notice the series of inclusions LU $\subset$ LU+anc $\subset$ LO $\subset$ LOCC for the above classes of operations, which implies, for instance, that LOCC simulation protocols contain all LU simulation protocols. Thus, it is to be expected that LOCC protocols are more efficient (in terms of the ration t'/t) than LU protocols.

2. Infinitesimal simulations

Notice that, in principle, the optimal —i.e. maximal— simulation factor $s(t') \equiv t'/t$ may depend on t'. However, we are ultimately interested in classifying and comparing the non-local properties of interaction Hamiltonians irrespectively of interaction times.

A sensible way of proceeding is by considering the worst case situation, namely the time t' for which the optimal ratio $s(t')$ achieves its minimal value. This occurs for an infinitesimal time t'. That is, simulations of H' for a time such that $||H't'|| << 1$ are, comparatively, the most expensive in terms of the required time t of interaction H. The reason is that, (*i*) simulations for an infinitesimal time are a particular case of simulation, providing an upper bound for the minimum of $s(t')$, and (*ii*) any finite-time simulation —or *gate simulation*— can be achieved, maybe not optimally, by concatenating infinitesimal-time simulations.

We shall denote $s_{H'|H}$ the limit $\lim_{t'\to 0} s(t')$, and call it the simulation factor of H' with H. Then, apart from quantifying the time overhead required in infinitesimal simulations, $s_{H'|H}$ has also two other meanings:

- $T'/s_{H'|H}$ upper bounds the time T of use of H needed to perform the unitary gate $e^{-iH'T'}$, for any T' (*gate simulation*);
- $s_{H'|H}$ is the time overhead required in *dynamics simulation*—i.e., $s_{H'|H}$ is the optimal ratio T'/T, where T is the time of H required to simulate the *entire* evolution of a system according to $e^{-it'H'}$, where t' runs from 0 to T'.

3. Optimal and efficient simulations

For any class C of the above operations and in the small time limit, the space of achievable evolutions using Hamiltonian H and operations C turns out to be convex. Then the following two problems,

P1: *Given any H and H', determine when H' can be efficiently (i.e., $t' = t$) simulated with H for infinitesimal times, denoted*

$$H' \prec_C H; \tag{1}$$

P2: *Given any H and H', determine the simulation factor $s_{H'|H}$;*

are equivalent, since $s_{H'|H}$ is nothing but the greatest s such that sH' can be efficiently simulated with H.

II. EQUIVALENCE OF LOCC AND LU+ANC PROTOCOLS

As mentioned, the solutions to problems P1 and P2 depend, in principle, on the class C of operations allowed during the simulation process. In this section we shall show that, for infinitesimal times, N-particle Hamiltonian simulation using LOCC is fully equivalent, as far as time efficiencies are concerned, to that using LU+anc. Thus, although LU+anc is just a small subset of LOCC, it turns out that both classes of operations lead to the same optimal solution for infinitesimal Hamiltonian simulations.

A. Hamiltonian simulation using LOCC

For simplicity sake we will perform most of the analysis in the simplest non-trivial case, that involving only two qubits, because this already contains all the ingredients of the general N-particle setting. Suppose, then,

that qubits A and B interact according to H for an overall time t, and that, simultaneously, they are being manipulated locally. This local manipulation corresponds to fast generalized measurements on each of the qubits. One such local measurement $\mathcal{M}$ on, say, qubit A, may imply appending a (possibly composite) auxiliary system or ancilla A' to A, performing unitary transformations and orthogonal measurements on AA', and disposing of (part of) the auxiliary system at wish. The state resulting from the measurement may involve not only the original qubit A but also (part of) the ancilla A'. In addition, the use of classical communication between the two qubits A and B allows the measurement $\mathcal{M}$ to depend on the result of previous measurements performed in either of the two sides of the composite system.

Thus, a general simulation strategy for time t (see figure 1) is characterized by a partition $\{t_1, t_2, ..., t_n\}$ of t, where $\sum_i t_i = t$, and a set of local measurements, $\{(\mathcal{M}_0, \mathcal{N}_0), (\mathcal{M}_1^{\alpha_1}, \mathcal{N}_1^{\alpha_1}), ..., (\mathcal{M}_n^{\alpha_n}, \mathcal{N}_n^{\alpha_n})\}$. The simulation strategy starts with measurements $\mathcal{M}_0$ and $\mathcal{N}_0$ being performed on AA' and BB', respectively. Then the two qubits A and B are left evolve according to H for a time t_1. After that, measurements $\mathcal{M}_1^{\alpha_1}$ and $\mathcal{N}_1^{\alpha_1}$ are performed. Here, index α_1 indicates that the measurements being performed after time t_1 may depend on the outcomes obtained from $\mathcal{M}_0$ and $\mathcal{N}_0$. Again, the measurements are followed by an evolution, for time t_2, of A and B according to H, and the protocol continues in an iterative fashion. In step k the two qubits A and B are first left evolve according to H for a time t_k and then measurements $\mathcal{M}_k^{\alpha_k}$ and $\mathcal{N}_k^{\alpha_k}$ (possibly depending on the outcome of all previous measurements) are locally performed in AA' and BB'. The protocol finishes after measurements $\mathcal{M}_n^{\alpha_n}$ and $\mathcal{N}_n^{\alpha_n}$ have been performed. Note that the protocol has a tree structure, starting with a preestablished couple of local manipulations and ending up at the extreme of a branch, characterized by the outcomes of all conditional local measurements performed up to $k = n$.

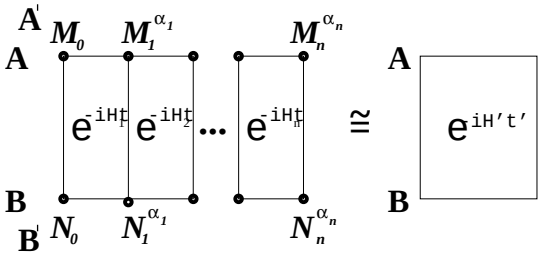

FIG. 1. Schematic representation of a Hamiltonian simulation protocol using LOCC. The unitary evolution of the composite system AB according to H and for a time $t = \sum_i t_i$ is interspersed with local measurements $\mathcal{M}_k^{\alpha_k}$ on systems AA' and $\mathcal{N}_k^{\alpha_k}$ on systems BB' to obtain a unitary evolution of AB according to H' and for a time t'.

Let us now focus on an arbitrary branch of the protocol, that we shall denote Γ. It is characterized by a series of measurement operators $\{(M_0^\Gamma, N_0^\Gamma), ..., (M_n^\Gamma, N_n^\Gamma)\}$ satisfying, for any vector $|\psi\rangle$ of the two-qubit system AB,

$$\sqrt{p_\Gamma}(e^{-iH't'}|\psi\rangle) \otimes (W_{A'B'}|0_{A'}0_{B'}\rangle = \left(M_n^\Gamma \otimes N_n^\Gamma e^{-it_nH} \cdots M_1^\Gamma \otimes N_1^\Gamma e^{-it_1H} M_0^\Gamma \otimes N_0^\Gamma\right)|\psi\rangle \otimes |0_{A'}0_{B'}\rangle. \quad (2)$$

Here operators M_k^Γ and N_k^Γ are associated, respectively, to the outcome of measurements $\mathcal{M}_k^{\alpha_k}$ and $\mathcal{N}_k^{\alpha_k}$ that characterize branch Γ. The ancillas A' and B', initially in some state $|0_{A'}\rangle$ and $|0_{B'}\rangle$, end up in the normalized state $W_{A'B'}|0_{A'}0_{B'}\rangle$. Notice that any branch corresponds in general to a trace-decreasing transformation, and that p_Γ is the probability that the protocol ends up following branch Γ. After the last step of the protocol the ancillas $A'B'$ are discarded. Then, the effective transformation for the two qubits AB is given by an operator which is proportional to the unitary transformation $e^{-iH't'}$.

B. LOCC-protocols are as efficient as LU+anc protocols for infinitesimal time simulations

As we have argued, we are interested here in Hamiltonian simulations for an infinitesimal simulation time t. In this regime Eq. (2) significantly simplifies, because we can expand the exponentials to first order in t, thereby obtaining an equation linear in H' and H. In addition, if t is small then qubits A and B only interact "a little bit". In what follows we will use this fact to prove the main result of this section, namely that all the measurement operators $\{M_i^\Gamma, N_i^\Gamma\}_{i=0}^n$ must be, up to negligible corrections, proportional to unitary operators in some corresponding relevant supports. What this will imply, in practice, is that LU+anc protocols can already simulate any $H't'$ achievable in a LOCC protocol.

1. LOCC-protocols for infinitesimal time simulations

For an infinitesimal time t we can rewrite Eq. (2), up to $\mathcal{O}(t^2)$ corrections and taking into account that it must hold for any $|\psi\rangle$, as

$$\sqrt{p_\Gamma}\,(I_{AB} - its H') \otimes (W_{A'B'}|0_{A'}0_{B'}\rangle) = \left(M_0 \otimes N_0 - it\sum_{k=1}^n p_k M_k \otimes N_k H M_k' \otimes N_k'\right)|0_{A'}0_{B'}\rangle, \quad (3)$$

where $s \equiv t'/t$, and where for $k = 1, \cdots, n$, we have introduced $p_k \equiv t_k/t$, $M_k \equiv M_n^\Gamma \cdots M_k^\Gamma$ and $M_k' \equiv M_{k-1}^\Gamma \cdots M_0^\Gamma$, and M_0 is defined as $M_n^\Gamma \cdots M_0^\Gamma$ (analogous definitions hold for operators N_k, N_k' and N_0). Eq. (3) implies, up to $\mathcal{O}(t)$ corrections,

$$\sqrt{p_\Gamma} I_{AB} \otimes (W_{A'B'}|0_{A'}0_{B'}\rangle) = M_0|0_{A'}\rangle \otimes N_0|0_{B'}\rangle, \quad (4)$$

which means, first, that $W_{A'B'}|0_{A'}0_{B'}\rangle$ is a product vector $|\phi_{A'}\rangle \otimes |\varphi_{B'}\rangle$ up to $\mathcal{O}(t)$ corrections. We shall assume

that $|\phi_{A'}\rangle \otimes |\varphi_{B'}\rangle$ is again the initial product vector of the ancillas $|0_{A'}0_{B'}\rangle$. [This can always be accomplished through an irrelevant, final local unitary in A' and B' —recall that systems A' and B' are to be traced out in the end.] With this assumption in mind we notice that Eq. (4) also implies that operators M_0 and N_0 in Eq. (3) must fulfill

$$M_0|0_{A'}\rangle = \sqrt{p_\Gamma} q I_{AA'}|0_{A'}\rangle + \mathcal{O}(t), \tag{5}$$
$$N_0|0_{B'}\rangle = \quad q^{-1} I_{BB'}|0_{B'}\rangle + \mathcal{O}(t), \tag{6}$$

which in turn means that, for all k,

$$M'_k|0_{A'}\rangle = \sqrt{p_\Gamma} q M_k^{-1}|0_{A'}\rangle + \mathcal{O}(t), \tag{7}$$
$$N'_k|0_{B'}\rangle = \quad q^{-1} N_k^{-1}|0_{B'}\rangle + \mathcal{O}(t). \tag{8}$$

By substitution in Eq. (3), and considering all contributions $\mathcal{O}(t)$, we find that the equation characterizing the achievable sH' in terms of the operators M_k and N_k reads

$$\begin{aligned} &sH' \otimes I_{A'B'}|0_{A'}0_{B'}\rangle = \\ &\sum_k p_k M_k \otimes N_k (H \otimes I_{A'B'}) M_k^{-1} \otimes N_k^{-1}|0_{A'}0_{B'}\rangle + \\ &(K_{AA'} \otimes I_{BB'} + I_{AA'} \otimes K'_{BB'} + I_{AB} \otimes K''_{A'B'})\,|0_{A'}0_{B'}\rangle, \end{aligned} \tag{9}$$

where the terms $K_{AA'}|0_{A'}\rangle$, $K'_{BB'}|0_{B'}\rangle$ and $K''_{A'B'}|0_{A'}0_{B'}\rangle$ account for the $\mathcal{O}(t)$ corrections of $M_0|0_{A'}\rangle$, $N_0|0_{B'}\rangle$ and $W_{A'B'}|0_{A'}0_{B'}\rangle$, respectively, and will turn out to be irrelevant.

2. Unitarity and entanglement conservation

Let us carry on by focusing our attention only on the operations performed on systems AA'. In order to show that, indeed, each operator $M_k|0_{A'}\rangle$ must be proportional to $U_k|0_{A'}\rangle$ for some unitary U_k acting on AA', we will use the fact that the protocol must be able to keep the entanglement of A with another system.

Let us suppose, thus, that qubit A is entangled with a distant qubit C, with the maximally entangled vector $|\Phi^+_{AC}\rangle \equiv (|0_A\rangle \otimes |0_C\rangle + |1_A\rangle \otimes |1_C\rangle)/\sqrt{2}$ describing the pure state of AC. Any unitary evolution of qubits A and B preserves the amount of entanglement between qubit C and qubits AB. In particular, qubit C must be still maximally entangled with AB after the simulated evolution $e^{-iH't'}$. This sets very strong restrictions on the kind of measurements that can be done during the simulation protocol. Let us consider, for instance, the first measurement operator M_0^Γ that acts on AA', with A' in state $|0_{A'}\rangle$ (see Eq. (2)). Without the above constraint we would have

$$M_0^\Gamma|0_{A'}\rangle = r_0\left(|\mu_{AA'}\rangle\langle 0_A| + s|\nu_{AA'}\rangle\langle 1_A|\right), \tag{10}$$

where the positive factors r_0 and s and the normalized vectors $|\mu_{AA'}\rangle$ and $|\nu_{AA'}\rangle$ fulfill $\langle 0_{A'}|M_0^{\Gamma\dagger}M_0^\Gamma|0_{A'}\rangle \leq I_A$, since M_0^Γ is a measurement operator. This measurement operator would transform vector $|\Phi^+_{AC}\rangle \otimes |0_{A'}\rangle$ into the (renormalized) vector

$$\frac{1}{\sqrt{1+s^2}}\left(|\mu_{AA'}\rangle \otimes |0_C\rangle + s|\nu_{AA'}\rangle \otimes |1_C\rangle\right), \tag{11}$$

which is still a maximally entangled state only if $s = 1$ and $|\mu^\perp_{AA'}\rangle \equiv |\nu_{AA'}\rangle$ is orthogonal to $|\mu_{AA'}\rangle$. That is, the entanglement with C is preserved only if $M_0^\Gamma|0_{A'}\rangle$ is proportional to $U|0_{A'}\rangle$, where U is some unitary transformation satisfying

$$U|0_{A'}\rangle = |\mu_{AA'}\rangle\langle 0_A| + |\mu^\perp_{AA'}\rangle\langle 1_A|. \tag{12}$$

Now, since a loss of this entanglement could not be recovered in subsequent steps of the simulation protocol —$AA'BB'$ cannot interact with the distant qubit C, and entanglement does not increase under LOCC—, we conclude that $M_0^\Gamma|0_{A'}\rangle$ must indeed be proportional to $U|0_{A'}\rangle$ or, equivalently, the operators M_1 and M'_1 in Eq. (3) necessarily fulfill (recalling also Eq. (7))

$$M'_1|0_{A'}\rangle = q_1 U_1^{-1}|0_{A'}\rangle, \tag{13}$$
$$\langle 0_{A'}|M_1 = q\sqrt{p_\Gamma} q_1^{-1}\langle 0_{A'}|U_1, \tag{14}$$

for some factor $q_1 \leq 1$ and some unitary U_1.

A similar argument can now be applied to operators M_k and M'_k of Eq. (3), for all k. Let us make it explicit for $k = 2$. We have $M'_2 = M_1^\Gamma M_0^\Gamma$, where $M_0^\Gamma|0_{A'}\rangle$ is proportional to $U|0_{A'}\rangle$, so that after measurement $\mathcal{M}_0$ the entanglement between systems AA' and C, in pure state $(|\mu_{AA'}\rangle\otimes|0_C\rangle+|\mu^\perp_{AA'}\rangle\otimes|1_C\rangle)/\sqrt{2}$, has been entirely preserved. Notice that the subsequent evolution e^{-iHt_1} is for infinitesimal time $t_1 = p_1 t$. This means that, up to corrections $\mathcal{O}(t)$, the state of $AA'C$ remains unchanged after time t_1. In other words, in order to preserve the entanglement with C we have that the relevant part of the measurement operator M_1^Γ acting on AA' has to be

$$M_1^\Gamma|\mu_{AA'}\rangle = r_1|\mu^1_{AA'}\rangle + \mathcal{O}(t), \tag{15}$$
$$M_1^\Gamma|\mu^\perp_{AA'}\rangle = r_1|\mu^{1\perp}_{AA'}\rangle + \mathcal{O}(t), \tag{16}$$

for a new couple $\{|\mu^1_{AA'}\rangle, |\mu^{1\perp}_{AA'}\rangle\}$ of orthonormal vectors and a positive $r_1 \leq 1$, so that the state of $AA'C$ after measurement $\mathcal{M}_1^{\alpha_1}$,

$$\frac{1}{\sqrt{2}}(|\mu^1_{AA'}\rangle \otimes |0_C\rangle + |\mu^{1\perp}_{AA'}\rangle \otimes |1_C\rangle) + \mathcal{O}(t), \tag{17}$$

is still maximally entangled with C. That is, the operator $M'_2|0_{A'}\rangle = M_1^\Gamma M_0^\Gamma|0_{A'}\rangle = q_2(|\mu^1_{AA'}\rangle\langle 0_A| + |\mu^{1\perp}_{AA'}\rangle\langle 1_A|) + \mathcal{O}(t)$ is also essentially proportional to $U_2^\dagger|0_{A'}\rangle$ for some unitary transformation U_2, whereas operator $\langle 0_{A'}|M_2$ is proportional to $\langle 0_{A'}|U_2$.

The above argument can also be applied to the measurement operators acting on BB'. Therefore we can replace operators M_k and N_k of Eq. (9) by unitary transformations, and neglect their $\mathcal{O}(t)$ corrections because

they contribute as $\mathcal{O}(t^2)$ in Eq. (9). In the last step of the protocol we trace out the ancillas. This corresponds to projecting Eq. (9) onto $\langle 0_{A'}0_{B'}|$, which leads to

$$sH' = \langle 0_{A'}0_{B'}| \sum_k p_k U_k \otimes V_k (H \otimes I_{A'B'}) U_k^\dagger \otimes V_k^\dagger |0_{A'}0_{B'}\rangle + M_A \otimes I_B + I_A \otimes M'_B + aI_{AB}, \quad (18)$$

for some hermitian operators M_A and M'_B and a real constant a. As discussed in [4], local terms such as M_A and M'_B can be always added and removed from sH' using LU transformations, so that we can dispose with them, and the term I_{AB} is also irrelevant for the dynamics of AB. Therefore, any Hamiltonian sH' that can be simulated with H using LOCC is locally equivalent to

$$\sum_k p_k \langle 0_{A'}0_{B'}|U_k \otimes V_k (H \otimes I_{A'B'}) U_k^\dagger \otimes V_k^\dagger |0_{A'}0_{B'}\rangle. \quad (19)$$

3. Equivalence between LOCC and LU+anc protocols

The set S of Hamiltonians that can be efficiently simulated with H is *convex*: if H can efficiently simulate H_1 and H_2, then it can also efficiently simulate $(H_1 + H_2)/2$. Indeed, we just need to divide the infinitesimal time t into two halves, and simulate H_1 for time $t/2$ and then H_2 for time $t/2$. The resulting Hamiltonian is precisely the previous average of H_1 and H_2. Thus, in order to characterize the convex set S, we can focus on its *extreme points*. Notice that the previous argument also holds for the set S' of Hamiltonians that can be efficiently simulated with LU+anc, so that S' also convex.

Now, Eq. (19) says that all points in S can be obtained as a convex combination of terms of the form

$$\langle 0_{A'}0_{B'}|U \otimes V (H \otimes I_{A'B'}) U^\dagger \otimes V^\dagger |0_{A'}0_{B'}\rangle. \quad (20)$$

In addition, in appendix A we show that any such a term can be obtained using LU+anc. It follows that any extreme point of S is of the form (20), so that it also belongs to S'. This finishes the proof of the fact that infinitesimal time simulations using LOCC can always be accomplished using LU+anc.

Summarizing, we have seen that any (rescaled) Hamiltonian sH' achievable in branch Γ of our LOCC-simulation protocol (cf. Eq. (19)) can also be achieved by just using local unitary transformations and ancillas as extra resources. It is now straightforward to generalize the above argument to N systems, each one having two or more levels, thereby extending the equivalence of LOCC and LU+anc protocols to general multiparticle interactions. Indeed, for any d-level system involved in the simulation, we just need to require that its entanglement with some remote, auxiliary d-level system be preserved, and we readily obtain that all measurement performed during the simulation protocol can be replaced with local unitary operations. We thus can conclude, using the notation introduced in section I, that

$$H' \prec_{LOCC} H \Leftrightarrow H' \prec_{LU+anc} H. \quad (21)$$

III. LU+ANC PROTOCOLS ARE NOT EQUIVALENT TO LU PROTOCOLS

The equivalence between infinitesimal-time simulations using LOCC and LU+anc may be conceived as a very satisfactory result. On the one hand, in order to, say, find the simulation factor $s_{H'|H}$, it is certainly easier to optimize over the set of simulation strategies that only use LU+anc, rather than over those strategies that use LO or even LOCC. On the other hand it is reassuring to see that, despite the diversity of classes of operations that we may use as a criterion to characterize the non-local properties of multiparticle interactions, most of these criteria (LOCC, LO and LU+anc) lead to the same classification and quantification. In other words, we do not have to deal with a large number of alternative characterizations. We shall show here, however, that simulation using only LU, that is, without ancillas, is not equivalent to that using LU+anc.

A. LU+anc protocols versus LU protocols

We have seen that the convex set of Hamiltonians that can be efficiently simulated with H using LU+anc (equivalently LOCC) contain elements of the form

$$\mathcal{E}(H) \equiv \langle 0_{A'}0_{B'}|U \otimes V (H \otimes I_{A'B'}) U^\dagger \otimes V^\dagger |0_{A'}0_{B'}\rangle \quad (22)$$

as extreme points. On the other hand if LU protocols were as powerful as LU+anc protocols, then we could always find a probability distribution $\{p_k\}$, $\sum_k p_k = 1$, local unitaries $\{u_k\}$ and $\{v_k\}$ acting on A and B, local self-adjoint, trace-less operators m and n and a real a such that the simulated $\mathcal{E}(H)$ satisfies

$$\mathcal{E}(H) = \sum_k p_k u_k \otimes v_k H u_k^\dagger \otimes v_k^\dagger + m \otimes I_B + I_A \otimes n + aI_{AB}, \quad (23)$$

so that $\mathcal{E}(H)$ can be expressed as a convex combination of local unitary conjugations of H plus some local terms and the identity operator. Recall that local terms can be added or subtracted by rotating the systems A and B infinitesimally [4], and that I_{AB} is physically irrelevant.

In [4] it was shown that, in the particular case of two-qubits, condition (23) can always be fulfilled. Next we shall show that this is not always the case for Hamiltonians of two d-level systems for $d > 2$, and also for Hamiltonians of more than two systems.

B. Inequivalence between LU+anc and LU protocols

1. Example 1: two d-level systems ($d > 2$)

Let us consider a d-level system A, with $d > 2$, a d-level ancilla A', and a unitary transformation U satisfying

$$\langle 0_{A'}|U = |0_A\rangle\langle 1_A 0_{A'}| + \sum_{i=1}^{d-1} |i_A\rangle\langle i_A i_{A'}|. \tag{24}$$

As we discuss in appendix A, the transformation of a Hamiltonian H acting on AB,

$$\mathcal{E}(H) \equiv \langle 0_{A'}0_{B'}|U(H \otimes I_{A'B'})U^\dagger|0_{A'}0_{B'}\rangle, \tag{25}$$

can be achieved using LU+anc [notice that this corresponds to choosing $V_{BB'} = I_{BB'}$ in eq. (A1)]. In particular, this transformation takes the two-particle Hamiltonian

$$K \equiv P_0 \otimes P_0 + \sum_{i=1}^{d-1} P_i \otimes P_i, \tag{26}$$

where $P_1 \otimes P_j \equiv |i_A\rangle\langle i_A| \otimes |j_B\rangle\langle j_B|$, into

$$\mathcal{E}(K) = P_0 \otimes P_1 + \sum_{i=1}^{d-1} P_i \otimes P_i. \tag{27}$$

If this simulation is to be possible using LU, then condition (23), which reads

$$\begin{aligned}\mathcal{E}(K) = &\sum_{i=0}^{d-1}\sum_k p_k u_k P_i u_k^\dagger \otimes v_k P_i v_k^\dagger \\ &+ m \otimes I_B + I_A \otimes n + aI_{AB},\end{aligned} \tag{28}$$

has to be fulfilled. Taking the trace of this expression we obtain $a = 0$. Now, tracing only system A leads to

$$2P_1 + \sum_{i=2}^{d-1} P_i = I + dn, \tag{29}$$

so that $n = (-P_0 + P_1)/d$. If instead we trace out system B we obtain

$$I = I + dm, \tag{30}$$

which means that operator m vanishes. Then, by noticing that $M \equiv \sum_{i=0}^{d-1}\sum_k p_k u_k P_i u_k^\dagger \otimes v_k P_i v_k^\dagger$ is a non-negative operator, we obtain that condition (23) can not be fulfilled because it leads to the following contradiction

$$\begin{aligned}0 = \text{tr}\,[P_0 \otimes P_1 \mathcal{E}(K)] = &\ \text{tr}\,[P_0 \otimes P_1 M] \\ + &\ \text{tr}\,[P_0 \otimes P_1 I_A/d \otimes (-P_0 + P_1)] \\ = &\ \text{tr}\,[P_0 \otimes P_1 M] + 1/d \geq 1/d.\end{aligned} \tag{31}$$

Thus, for any $d > 2$, we have explicitly constructed an example of LU+anc simulation for Hamiltonians acting on two d-level systems that can not be achieved using only LU. We note, however, that for two-particle Hamiltonians, LU+anc and LU protocols only differ quantitatively, for LU protocols are able to simulate any bipartite Hamiltonian H' starting from any other H with non-vanishing $s_{H'|H}$ [6].

2. Example 2: a $2 \times 2 \times 2$ composite system

Let us consider now the simulation, for an infinitesimal time t, of the three-qubit Hamiltonian $I \otimes \sigma_3 \otimes \sigma_3$ using $\sigma_3 \otimes \sigma_3 \otimes \sigma_3$, where

$$\sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{32}$$

This is possible, when allowing for LU+anc operations, by considering the transformation U acting on qubit A and on a qubit-ancilla A' in state $|0_{A'}\rangle$, where

$$\langle 0_{A'}|U = |0_A\rangle\langle 0_A| \otimes \langle 0_{A'}| + |1_A\rangle\langle 0_A| \otimes \langle 1_{A'}|, \tag{33}$$

Indeed, we have that $\langle 0_{A'}|U(\sigma_3 \otimes I_{A'})U^\dagger|0_{A'}\rangle = I_A$. On the other hand it is impossible to simulate $I \otimes \sigma_3 \otimes \sigma_3$ using $\sigma_3 \otimes \sigma_3 \otimes \sigma_3$, for it would imply to transform σ_3 into I through unitary mixing, which is a trace-preserving operation. It is straightforward to construct similar examples in higher dimensional systems, and also with more than three systems.

We note that, as far as interactions involving more than two systems are concerned, the inequivalence between LU+anc and LU simulation protocols is not only quantitative, leading to different simulation factors, but also qualitative. The example above shows that LU protocols can simply not be used to simulate Hamiltonians that can be simulated using LU+anc and the same interaction H, irrespective of time considerations. The reason for this can be understood by noticing the following. Unlike in the bipartite setting, when say three parties are involved, a Hamiltonian proportional to the identity for one of the parties, such as $I_A \otimes K_B \otimes K'_C$, is in general no longer a local Hamiltonian. That is, Hamiltonian terms proportional to the identity in one system may correspond to a non-local Hamiltonian for the rest of the systems. Such terms cannot be added or subtracted at wish using only LU, as it happened with terms such as $I_A \otimes K_B$ in the case of two systems.

IV. OPTIMAL SIMULATION OF TWO-QUBIT HAMILTONIANS USING LOCC

In this last section we address the problem of optimal Hamiltonian simulation using LU for the case of two-qubit interactions. We recover the results of [4], but through an alternative, simpler proof, based on known results of majorization theory —and thus avoiding the

geometrical constructions of the original derivation [4]. The equivalence of LOCC and LU+anc strategies presented in section II together with that of LU+anc and LU strategies for two-qubit Hamiltonians proved in [4] imply that these results are also optimal in the context of LOCC Hamiltonian simulation.

We start by recalling some basic facts. Any two-qubit Hamiltonian H is equivalent, as far as LU simulation protocols are concerned, to its canonical form [1,4]

$$H = \sum_{i=1}^{3} h_i \sigma_i \otimes \sigma_i, \tag{34}$$

where $h_1 \geq h_2 \geq |h_3| \geq 0$ and the operators σ_i are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{35}$$

Let $\{|\Phi_i\rangle\}$ stand for the basis of maximally entangled vectors of two qubits

$$|\Phi_1\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Phi_2\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$
$$|\Phi_3\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad |\Phi_4\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{36}$$

Then H can be alternatively expressed as

$$H = \sum_{i=1}^{4} \lambda_i |\Phi_i\rangle\langle\Phi_i|, \tag{37}$$

where λ_i are decreasingly ordered, real coefficients fulfilling the constraint $\sum_i \lambda_i = 0$ (coming from the fact that H has no trace) and

$$\lambda_1 = h_1 + h_2 - h_3 \tag{38}$$
$$\lambda_2 = h_1 - h_2 + h_3 \tag{39}$$
$$\lambda_3 = -h_1 + h_2 + h_3 \tag{40}$$
$$\lambda_4 = -h_1 - h_2 - h_3. \tag{41}$$

The most general simulation protocol using H and LU leads to

$$H' = \sum_k p_k u_k \otimes v_k H u_k^\dagger \otimes v_k^\dagger, \tag{42}$$

where we have assumed, without loss of generality, that H' is also in its canonical form, as in Eqs. (34) and (37), with corresponding coefficients h_i' and λ_i'.

A. Necessary and sufficient conditions for efficient simulation and optimal simulation factor

Let us derive the necessary and sufficient conditions for H to be able to simulate H' using LU and for infinitesimal simulation times. Uhlmann's theorem [7] states that the eigenvalues λ_i' of operator H' in Eq. (42), a unitary mixing of operator H, are majorized by the eigenvalues λ_i of H, that is

$$\begin{aligned} \lambda_1' &\leq \lambda_1, \\ \lambda_1' + \lambda_2' &\leq \lambda_1 + \lambda_2, \\ \lambda_1' + \lambda_2' + \lambda_3' &\leq \lambda_1 + \lambda_2 + \lambda_3, \\ \lambda_1' + \lambda_2' + \lambda_3' + \lambda_4' &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \end{aligned} \tag{43}$$

where the last equation is trivially fulfilled due to the fact that H and H' are trace-less operators. Succinctly, we shall write $\vec{\lambda'} \prec \vec{\lambda}$, as usual [8]. In terms of the coefficients h_i and h_i' the previous conditions read

$$\begin{aligned} h_1' &\leq h_1, \\ h_1' + h_2' - h_3' &\leq h_1 + h_2 - h_3, \\ h_1' + h_2' + h_3' &\leq h_1 + h_2 + h_3, \end{aligned} \tag{44}$$

and correspond to the s(pecial)-majorization relation, $\vec{h'} \prec_s \vec{h}$, introduced in Ref. [4]. Thus, we have already recovered the necessary conditions [4] for H to be able to *efficiently* simulate H' in LU protocols [9] (and thus, since we are in the two-qubit case, also in LOCC protocols).

In order to see that conditions (43) [and thus conditions (44)] are also sufficient for efficient LU simulation, we concatenate two other results of majorization theory. The first one (see theorem II.1.10 of [8]) states that $\vec{\lambda'} \prec \vec{\lambda}$ if and only if a doubly stochastic matrix m exists such that $\lambda_i' = \sum_j m_{ij} \lambda_j$. The second result is known as Birkhoff's theorem [8], and states that the matrix m can always be written as a convex sum of permutation operators $\{P_k\}$, so that

$$\begin{pmatrix} h_1' \\ h_2' \\ h_3' \\ h_4' \end{pmatrix} = \sum_k p_k P_k \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}. \tag{45}$$

This means that whenever conditions (43) are fulfilled we can obtain H' from H by using a mixing of unitary operations T_i, where each T_i permutes the vectors $\{|\Phi_i\rangle\}$,

$$H' = \sum_i p_i T_i H T_i^\dagger. \tag{46}$$

Then, all we still need to see is that all $4! = 24$ possible permutations of the vectors $\{|\Phi_i\rangle\}$ can be performed through *local* unitaries T_i. Recall, however, that any permutation σ, taking elements $(1,2,3,4)$ into $(\sigma(1),\sigma(2),\sigma(3),\sigma(4))$, can be obtained by composing (several times) the following three transpositions,

$$(1,2,3,4) \to (2,1,3,4), \tag{47}$$
$$(1,2,3,4) \to (1,3,2,4), \tag{48}$$
$$(1,2,3,4) \to (1,2,4,3), \tag{49}$$

where each permutation affects two neighboring elements. The corresponding three basic permutations of $(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ can be easily obtained using LU. Indeed, in order to permute $(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ into

$$\begin{aligned}&(\Phi_2, \Phi_1, \Phi_3, \Phi_4),\\ &(\Phi_1, \Phi_3, \Phi_2, \Phi_4),\\ &(\Phi_1, \Phi_2, \Phi_4, \Phi_3),\end{aligned} \tag{50}$$

we can simply apply, respectively, the following local unitaries:

$$\begin{aligned}&\frac{I - i\sigma_1}{\sqrt{2}} \otimes \frac{I - i\sigma_1}{\sqrt{2}},\\ &\frac{I + i\sigma_3}{\sqrt{2}} \otimes \frac{I - i\sigma_3}{\sqrt{2}},\\ &\frac{I + i\sigma_1}{\sqrt{2}} \otimes \frac{I - i\sigma_1}{\sqrt{2}}.\end{aligned} \tag{51}$$

Therefore, any permutation σ of the states (36) can be accomplished through local unitaries T_i, and any Hamiltonian H' satisfying conditions (44) [equivalently, conditions (43)] can be efficiently simulated with H and LU.

In the following we condense the previous findings into two results, R1 and R2, which provide an explicit answer to problems P1 and P2, respectively. We assume that the two-qubit Hamiltonians H and H' are in their canonical form, with $\vec{\lambda}$, $\vec{h}$, $\vec{\lambda'}$ and $\vec{h'}$ the corresponding vectors of coefficients.

R1: *Hamiltonian H' can be efficiently simulated with H and LOCC —or LU, LU+anc, or LO— if and only if conditions (44) [or, equivalently, conditions (43)] are fulfilled, i.e.*

$$H' \prec_{LOCC} H \quad \Leftrightarrow \quad \vec{h'} \prec_s \vec{h} \quad \Leftrightarrow \quad \vec{\lambda'} \prec \vec{\lambda}. \tag{52}$$

R2: *The simulation factor $s_{H'|H}$ for LOCC —or LU, LU+anc, or LO— protocols is given by the maximal $s > 0$ such that $s\vec{h'} \prec_s \vec{h}$ or, equivalently, such that $s\vec{\lambda'} \prec \vec{\lambda}$.*

B. Explicit optimal LU protocols

The last question we would like to address is how to actually construct a simulation protocol. That is, given H and H', we would like to know how to simulate sH' using H and LU, for any $s \in [0, s_{H'|H}]$.

A complete answer to this question is given by a probability distribution $\{p_k\}$ and a set of unitaries $\{u_k \otimes v_k\}$ such that

$$sH' = \sum_k p_k u_k \otimes v_k H u_k^\dagger \otimes v_k^\dagger, \tag{53}$$

where $s \in [0, s_{H'|H}]$, and $s_{H'|H}$ can be obtained using result R2.

We already argued that it is always possible to choose all $u_k \otimes v_k$ such that they permute the vectors of Eq. (36), so that each $u_k \otimes v_k \equiv T_k$ is just a composition of the local unitaries of Eqs. (51). As before, let $\{P_k\}_{k=1}^{24}$ denote the 24 permutations implemented by the local unitaries $\{T_k\}_{k=1}^{24}$. Then the above problem reduces to finding an explicit probability distribution $\{p_k\}$ such that

$$s_{H'|H} H' = \sum_k p_k T_k H T_k^\dagger, \tag{54}$$

or, equivalently, such that

$$s_{H'|H} \vec{\lambda'} = \sum_k p_k P_k \vec{\lambda}. \tag{55}$$

This is done on appendix B using standard techniques of convex set theory. There we show how to construct a solution involving at most 4 terms $p_k T_k$ for $s < s_{H'|H}$, and at most 3 terms for optimal simulation, that is, when $s = s_{H'|H}$.

V. CONCLUSIONS

In this paper we have studied Hamiltonian simulation under the broader scope of LOCC protocols. We have focused on infinitesimal-time simulations, for which we have shown that LOCC protocols are equivalent to LU+anc protocols, also that LU+anc protocols are in general inequivalent to LU protocols (two-qubit Hamiltonians being an exception). For two-qubit Hamiltonians we have rederived and extended the results of [4], to finally provide the optimal solution using LOCC.

Thus, the problem of simulating Hamiltonian evolutions has received a complete answer for infinitesimal times and using LOCC, for the simplest case of two-qubit systems. Several interesting questions remain open. On the one hand, the generalization of these results to systems other than two qubits appears as challenging. On the other hand, the asymptotic scenario for Hamiltonian simulation, where H is used to simulate H' many times on different systems, certainly deserves a lot of attention. Finally, in [10] we will report the use of entanglement to catalyze the simulation of both infinitesimal-time and finite-time evolutions.

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APPENDIX A: LU+ANC SIMULATIONS AND DEPOLARIZATION

In this appendix we show that in LU+anc Hamiltonian simulation any Hamiltonian of the form

$$H' = \langle 0_{A'}0_{B'}|U\otimes V(H\otimes I_{A'B'})U^\dagger\otimes V^\dagger|0_{A'}0_{B'}\rangle \quad \text{(A1)}$$

can be efficiently simulated with H, for any couple of unitaries U and V acting on AA' and BB'. The result is valid also for more than two systems after a straightforward generalization of the following proof.

Notice that we can always write U and V using product basis $\{|i_Aj_{A'}\rangle\}$ and $\{|i_Bj_{B'}\rangle\}$ as

$$U = \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_{A'}-1} |i_Aj_{A'}\rangle\langle\phi_{ij}| \quad \text{(A2)}$$

$$V = \sum_{i=0}^{d_B-1}\sum_{j=0}^{d_{B'}-1} |i_Bj_{B'}\rangle\langle\psi_{ij}|, \quad \text{(A3)}$$

where $\{|\phi_{ij}\rangle\}$ and $\{|\psi_{ij}\rangle\}$ are other orthonormal basis of systems AA' and BB', respectively, and d_κ denotes the dimension of system κ. In order to simulate H' as defined in Eq. (A1) with H, we need to consider a series of local unitaries $\{U_a\otimes V_b\}$,

$$U_a \equiv I\otimes(\sum_{l=0}^{d_{A'}-1} e^{i2\pi\frac{al}{d_{A'}}}|l_{A'}\rangle\langle l_{A'}|)U, \quad \text{(A4)}$$

$$V_b \equiv I\otimes(\sum_{l=0}^{d_{B'}-1} e^{i2\pi\frac{bl}{d_{B'}}}|l_{B'}\rangle\langle l_{B'}|)V, \quad \text{(A5)}$$

and a constant probability distribution $\{p_{ab}\}$, $p_{ab} = 1/(d_{A'}d_{B'})$. Then we have that $U_a^\dagger|0_{A'}\rangle = U^\dagger|0_{A'}\rangle$, and that $\sum_a U_a = d_{A'}|0_{A'}\rangle\langle 0_{A'}|U$, and similarly for V_b, so that we obtain

$$\sum_{ab} p_{ab}U_a\otimes V_b(H\otimes I_{A'B'})U_a^\dagger\otimes V_b^\dagger|0_{A'}0_{B'}\rangle =$$

$$|0_{A'}0_{B'}\rangle\langle 0_{A'}0_{B'}|U\otimes V(H\otimes I_{A'B'})U^\dagger\otimes V^\dagger|0_{A'}0_{B'}\rangle. \quad \text{(A6)}$$

Therefore Eq. (A6) defines a protocol that simulates the Hamiltonain of Eq. (A1).

APPENDIX B: EXPLICIT TWO-QUBIT LU SIMULATION PROTOCOLS

In this appendix we show how to find a probability distribution $\{p_k\}$ and permutations $\{P_k\}$ such that

$$\vec{\mu} = \sum_k p_k P_k\vec{\lambda}, \quad \text{(B1)}$$

for any two given four-dimensional, real vectors $\vec{\lambda}$ and $\vec{\mu}$ ($\vec{\mu} = s\vec{\lambda'}$ in section IV.B) such that $\vec{\mu}\prec\vec{\lambda}$, where $\sum_{i=1}^4\lambda_i = \sum_{i=1}^4\mu_i = 0$.

We first note two facts that will allow us to use standard techniques of convex set theory: (i) the set $S\equiv\{\vec{\tau}\mid\vec{\tau}\prec\vec{\lambda}\}$ is convex, and (ii) $\{P_k\vec{\lambda}\}_{i=1}^{24}$ are the extreme points of S, as it follows from Birkhoff's theorem [8]. We can then proceed as follows.

Step (a): we check whether $\vec{\mu} = P_i\vec{\lambda}$ for any $i = 1,\cdots,24$. If we find one such permutation we are done. Otherwise we move to step (b).

Step (b): Facts (i) and (ii) warrantie that there is at least one permutation P_k, that we call Q_1, and a positive $\epsilon>0$ such that

$$\vec{\mu} = \epsilon Q_1\vec{\lambda} + (1-\epsilon)\vec{\tau}, \quad \text{(B2)}$$

where $\vec{\tau}$ also belongs to S, and therefore satisfies $\vec{\tau}\prec\vec{\lambda}$. In other words, we have to search until we find a permutation Q_1 such that

$$(\vec{\mu}-\epsilon Q_1\vec{\lambda})/(1-\epsilon)\prec\vec{\lambda}, \quad \text{(B3)}$$

for some $\epsilon>0$. Once we have found it we only need to increase ϵ to its maximal value compatible with Eq. (B3). Let q_1 be this maximal value of ϵ. Then we can write

$$\vec{\mu} = q_1Q_1\vec{\lambda} + (1-q_1)\vec{\mu}_2, \quad \text{(B4)}$$

where $\vec{\mu}_2\prec\vec{\lambda}$ is on one of the surfaces of S —otherwise we could have taken a greater q_1.

Such a surface is, again, a (lower dimensional) convex set, whose extreme points are some of the $P_k\vec{\lambda}$, and whose elements $\vec{\tau}$ fulfill $\vec{\tau}\prec\vec{\lambda}$ but with one of the majorization inequalities replaced with an equality. This allows as to repeat points (a) and (b), but in order to decompose $\vec{\mu}_2$ as a convex sum of vectors $P_k\vec{\lambda}$. That is, first we check whether $\vec{\mu}_2$ corresponds to $P_k\vec{\lambda}$ for some k. And, if not, we search until we find a permutation P_k, let us called it Q_2, such that, again,

$$(\vec{\mu}_2-\epsilon Q_2\vec{\lambda})/(1-\epsilon)\prec\vec{\lambda}. \quad \text{(B5)}$$

The maximum value of ϵ compatible with this equation, say q, leads to a second term q_2Q_2 ($q_2=(1-q_1)q$) for the decomposition of $\vec{\mu}$,

$$\vec{\mu} = (q_1Q_1+q_2Q_2)\vec{\lambda} + (1-q_1-q_2)\vec{\mu}_3, \quad \text{(B6)}$$

and to a new $\vec{\mu}_3$, that lies on a surface of still lower dimensionality of the original convex set S. We iterate the procedure until the remaining vector $\vec{\mu}_l$ lies on a convex surface of S of dimension zero, which means that the surface contains only one element, $\vec{\mu}_l$. In this way we obtain the wished decomposition,

$$\vec{\mu} = \sum_{k=1}^{l} q_kQ_k\vec{\lambda}. \quad \text{(B7)}$$

What is the minimal value of l? For non-optimal simulation protocols we have that $\vec{\mu} = s\vec{\lambda'}$, where $s<s_{H'|H}$,

and $\vec{\mu}$ is in the interior of S, which is a three-dimensional set. Therefore the above procedure has to be iterated at most three times before we are left with a zero-dimensional surface of S, so that a minimal decomposition contains at most with $l = 4$ terms. For optimal simulation protocols $\vec{\mu} = s_{H'|H}\vec{\lambda}'$ is already in a surface of S, and therefore the minimal decomposition contains, at most, from 1 to 3 terms.

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